PMM U.S.S.R.,Vol.45,pp.192-198
0021-8928/82/2 0192 $7.50/0

UDC 532.592

CONSTRUCTION OF THE FUNDAMENTAL SOLUTION FOR THE OPERATOR OF INTERNAL WAVES*

S.Ia. SEKERZH-ZEN'KOVICH

Investigation of the differential operator, to which many problems of the linear theory of waves in stratified fluid are reduced, is considered. A singularity of this operator is that in it the higher time derivative appears in the same term as the higher derivatives with respect to coordinates.

An operator of the considered here type appeared for the first time in the equation derive by Sobolev /1/ in the course of investigation of unsteady motions of a rotating fluid. Some extensions of Sobolev's equation were considered in /2–4/ and other publications.

Below, we use the theory of generalized functions for constructing the fundamental solution of the internal wave operator, and explain its hydrodynamic meaning. A summary of obtained results appeared in /5/.

1. Statement of the problem. The Fourier transform of the fundamental solution.

Consider the operator

$$\mathrm{N} = \frac{\partial^2}{\partial t^2}\Delta_3 + N^2\Delta_2 \tag{1.1}$$

that defines in linear and the Boussinesq approximations the process of internal wave propagation in a continuously stratified fluid. In this formula t is the time, Δ_3 is a three dimensional Laplace operator of space coordinates x_1, x_2, x_3, Δ_2 is a two dimensional Laplace operator of the horizontal coordinates x_1 and x_2, N is the so-called Brunt–Väisälä frequency which defines density distribution in an inhomogeneous fluid. It is usually assumed that the fluid density ρ_0 in the unperturbed state depends only on the vertical coordinate x_3, then, when the x_3-axis is directed against the gravity acceleration g, $N^2 = -g/\rho_0 d\rho_0/dx_3$. Below, we assume that $N^2 = \text{const} > 0$. This corresponds to steady exponential stratification and closely conforms with laboratory experiments. Operator N that corresponds to such stratification will be called the operator of internal waves.

Let us derive the fundamental solution $E(x, t)$ of the internal wave operator. By definition $E(x, t)$ is a generalized function /6/ that satisfies the equation

$$\mathrm{N}E = \delta(x, t) \tag{1.2}$$

in whose right-hand side we have the Dirac δ-function, and $x = (x_1, x_2, x_3)$ is a point in the three-dimensional Euclidean space R^3.

Restricting our investigation to the space $S'(R^4)$ of slow growing generalized functions, we shall use for the construction of $E(x, t)$ the method of the Fourier transform F_x in space variables x_j.

Applying F_x to Eq.(1.2), we obtain for the Fourier transform $E^*(\xi, t)$ of function $E(x, t)$ the equation

$$-|\xi|^2\frac{\partial^2 E^*}{\partial t^2} - N^2(\xi_1^2+\xi_2^2)E^* = 1(\xi)\delta(t), \quad |\xi| = (\xi_1^2+\xi_2^2+\xi_3^2)^{1/2}$$

one of whose solutions in S' is the function

$$E^*(\xi, t) = -\frac{\theta(t)}{|\xi|^2}\frac{\sin[\nu(\xi)t]}{\nu(\xi)}, \quad \nu(\xi) = \frac{N\sqrt{\xi_1^2+\xi_2^2}}{|\xi|}$$

where $\theta(t)$ is the Heaviside unit function.

Locally integrable functions and the generated by them in conformity with conventional rule generalized functions are identified in the last formula.

We call the generalized function

$$E(x, t) = F_\xi^{-1}[E^*]$$

the fundamental solution of the internal wave operator.

For the determination of $F_\xi^{-1}[E^*]$ we proceed as follows. We introduce the set of auxiliary functions $E_\gamma^*(\xi, t)$ that contain the positive parameter γ and whose property is that in

*Prikl.Matem.Mekhan.,45,No.2,266-274,1981

S', $E_\gamma^*(\xi, t) \to E^*(\xi, t)$ as $\gamma \to 0$. We determine $F_\xi^{-1}[E_\gamma^*]$ and then calculate the transform $F_\xi^{-1}[E^*]$ using the passing to limit and the continuity of the Fourier transform in space S'. The derived fundamental solution proves to be a regular generalized function.

2. Introduction of functions $E_\gamma^*(\xi, t)$. We introduce the set of functions

$$E_\gamma^*(\xi, t) = E^*(\xi, t)\exp(-\gamma|\xi|) \tag{2.1}$$

where γ is a positive parameter.

It can be shown that in S'

$$\lim E_\gamma^*(\xi, t) = E^*(\xi, t),\ \gamma \to +0 \tag{2.2}$$

For each basic function $\varphi(\xi, t) \in S$ and any positive number C we have

$$|(E^*, \varphi) - (E_\gamma^*, \varphi)| \leqslant I_1 + I_2, \quad I_1 = \Big|\int_{|\xi|\leqslant C} |\xi|^{-2} d\xi \int_0^\infty G(\xi, t, \gamma)\, dt\Big|, \quad I_2 = \Big|\int_{|\xi|>C} |\xi|^{-2} d\xi \int_0^\infty G(\xi, t, \gamma)\, dt\Big| \tag{2.3}$$

$$G(\xi, t, \gamma) = [1 - \exp(-\gamma|\xi|)]\frac{\sin\nu(\xi)t}{\nu(\xi)}\varphi(\xi, t)$$

Let ε be an arbitrary positive number. We fix $\varphi(\xi, t)$. Since $\varphi \in S$, it is possible to indicate a number $C > 0$ such that

$$C^{-2}\int_{R^4} |t\varphi(\xi, t)|\, d\xi\, dt < \frac{\varepsilon}{2}$$

Then

$$I_2 \leqslant C^{-2}\int_{|\xi|>C} d\xi \int_0^\infty \Big|\frac{\sin\nu(\xi)t}{\nu(\xi)t}\Big| \cdot |t\varphi(\xi, t)|\, dt \leqslant \frac{\varepsilon}{2}$$

We select $\gamma_0 > 0$ so that the inequality

$$[1 - \exp(-\gamma C)]\int_{R^4} |\xi|^{-2} |t\varphi(\xi, t)|\, d\xi\, dt < \frac{\varepsilon}{2}$$

is satisfied for $\gamma \in (0, \gamma_0)$.

Then for $\gamma \in (0, \gamma_0)$

$$I_1 \leqslant [1 - \exp(-\gamma_0 C)]\int_{|\xi|\leqslant C} |\xi|^{-2} d\xi \int_0^\infty |t\varphi(\xi, t)|\, dt < \frac{\varepsilon}{2}$$

and, consequently, taking into account (2.3), we obtain $|(E^*, \varphi) - (E_\gamma^*, \varphi)| < \varepsilon$.

3. Functions $E_\gamma(x, t)$ and their majorants. Let us determine the inverse Fourier transform

$$E_\gamma(x, t) = F_\xi^{-1}[E_\gamma^*]$$

By virtue of the absolute integrability of $E_\gamma^*(\xi, t)$ we have

$$E_\gamma = (2\pi)^{-3}\int_{R^3} E^* \exp[-\gamma|\xi| - i(x, \xi)]\, d\xi, \qquad (x, \xi) = x_1\xi_1 + x_2\xi_2 + x_3\xi_3$$

Substituting for function E^* its expression (2.1), passing to spherical coordinates β, φ, θ, and taking into account the integrand periodicity with respect to φ, we obtain

$$E_\gamma = -\frac{\theta(t)}{(2\pi)^3 N}\int_0^\pi \sin(Nt\sin\theta)\, d\theta \int_0^{2\pi} d\varphi \int_0^\infty \exp(-\beta H_1)\, d\beta$$

$$H_1 = \gamma + i(r\sin\theta\sin\varphi + x_3\cos\theta),\ r = (x_1^2 + x_2^2)^{1/2}$$

Integration with respect to β yields

$$E_\gamma(x, t) = -\frac{\theta(t)}{(2\pi)^3 N}\int_0^{\pi/2} \sin(Nt\sin\theta)\, d\theta \int_0^{2\pi} (H_1^{-1} + H_2^{-1})\, d\varphi \tag{3.1}$$

where function $H_2(\varphi, \theta; r, x_3)$ differs from H_1 only by the sign at $x_3 \cos\theta$.

Assuming that the inequality

$$r \neq 0 \tag{3.2}$$

holds, we transform formula (3.1) in two ways.

First, we substitute integration over the circle $|z| = 1$ in the plane of the complex variable z for integration with respect to φ. Then

$$E_\gamma = \frac{\theta(t)\, i}{4\pi^3 N r} \int_0^{\pi/2} \frac{\sin(Nt\sin\theta)}{\sin\theta} d\theta \int_{|z|=1} (Z_0^{-1} + Z_1^{-1})\, dz$$

$$Z_j(z; \varkappa, \gamma') = z^2 + 2\,[\gamma' + i\,(-1)^j \varkappa]\, z - 1; \quad j = 0, 1, \quad \gamma' = \frac{\gamma}{r\sin\theta}, \quad \varkappa = \frac{x_3}{r}\operatorname{ctg}\theta$$

Calculation of the internal integral using the theory of residues yields

$$E_\gamma = -\frac{\theta(t)}{2\pi^2 N r} \int_0^{\pi/2} \frac{\sin(Nt\sin\theta)}{\sin\theta} \left(\frac{1}{z_0 - z_2} + \frac{1}{z_1 - z_3)}\right) d\theta \tag{3.3}$$

where z_0, z_2 and z_1, z_3 are pairs of roots of equations

$$Z_0(z, \varkappa, \gamma') = 0, \quad Z_1(z; \varkappa, \gamma') = 0$$

respectively, and $|z_0| < 1$ and $|z_1| < 1$.

We introduce the notation

$$A = 1 + \gamma'^2 - \varkappa^2, \; B = 2\varkappa\gamma'$$

Then for roots z_0 and z_1 we have

$$z_j = -\gamma' - i\,(-1)^j \varkappa + (A^2 + B^2)^{1/4} \exp\left\{\frac{i}{2}(-1)^j \times \left[\operatorname{arctg}\frac{B}{A} + \frac{\pi}{2}(1 - \operatorname{sgn} A)\right]\right\}$$

and similar expressions but with the opposite sign at the term $(A^2 + B^2)^{1/4}$ for roots z_2 and z_3.

Substituting these expressions for roots z_j into (3.3) and passing to integration with respect to $u = \sin\theta$, after simple transformations, we obtain

$$E_\gamma(x, t) = -\frac{\sqrt{2}\,\theta(t)}{4\pi^2 N |x|} \int_0^1 H(u) \sin Ntu \frac{du}{\sqrt{1 - u^2}} \tag{3.4}$$

$$H(u) = P^{-1}(P + K)^{1/2}, \; K(u) = u^2 + (\gamma^2 - x_3^2) / |x|^2$$

$$P(u) = (K^2 + L^2)^{1/2}, \; L(u) = 2\gamma x_3 |x|^{-2} \sqrt{1 - u^2}$$

Let us now transform formula (3.1) in another way. Substituting the variable $\tau = \sin\varphi$ for the variable of integration φ, we obtain

$$E_\gamma(x, t) = -\frac{i\theta(t)}{4\pi^3 N r} \int_0^{\pi/2} \frac{\sin(Nt\sin\theta)}{\sin\theta} d\theta \times \int_{-1}^{1} [(\tau - \varkappa + i\gamma')^{-1} + (\tau + \varkappa + i\gamma')^{-1}] \frac{d\tau}{\sqrt{1 - \tau^2}} \tag{3.5}$$

Taking into account the inequality

$$|\tau \mp \varkappa + i\gamma'| \geqslant \gamma'$$

we conclude on the basis of the last formula that

$$|E_\gamma(x, t)| \leqslant (8\pi N\gamma)^{-1}, \; r \neq 0, \; \gamma \neq 0$$

i.e. $E_\gamma(x, t)$ are bounded locally integrable functions when $\gamma \neq 0$ and condition (3.2) are satisfied.

Let us transform formula (3.5). Carrying in it integration with respect to τ, for instance, using the theory of residues, we finally obtain

$$E_\gamma(x, t) = \frac{\theta(t)}{4\pi^3 N r} \int_0^{\pi/2} \frac{\sin(Nt\sin\theta)}{\sin\theta} \{[1 - (\varkappa - i\gamma')^2]^{1/2} + [1 - (-\varkappa - i\gamma')^2]^{1/2}\}\, d\theta \tag{3.6}$$

where the branch of root $\sqrt{1-\omega^2}$ has been chosen as follows: a slit has been affected in the ω-plane between points $\omega=-1$ and $\omega=1$, and it assumed that

$$\sqrt{1-\omega^2}=\sqrt{2} \quad \text{when} \quad \omega=i$$

It can be shown that functions $E_\gamma(x,t)$ have a local integrable majorant.

Let us, first, assume that $\gamma \geqslant |x_3|$. Then, taking into account that for real a and b

$$|(a+ib)^{1/2}| \geqslant |a|^{1/2}$$

we obtain from (3.6)

$$|E_\gamma(x,t)| \leqslant \frac{t}{2\pi^2 r}\int_0^{\pi/2} |1-\varkappa^2+\gamma'^2|^{-1/2} d\theta = \frac{t}{2\pi^2}\int_0^{\pi/2} (|x|^2\sin^2\theta+\gamma^2-x_3^2)^{-1/2}\sin\theta\, d\theta \leqslant \frac{t}{4\pi|x|}$$

Let now $\gamma<|x_3|$. We split the integration interval with respect to θ in formulas (3.5) and (3.6) in three intervals $[l_j, l_{j+1}]$, where $j=0,1,2$; $l_0=0$, $l_3=\pi/2$, $l_1=\arcsin m_1$, $l_2=\arcsin m_2$ $m_1=\sqrt{x_3^2-\gamma^2}/|x|$, $m_2=|x_3|/|x|$, and represent the respective integrals as the sum of three integrals J_1, J_2, and J_3. Let us estimate each of them.

For estimating integrals J_1 and J_3 we use formula (3.6). We have

$$|J_1| \leqslant \frac{t}{2\pi^2}\int_0^{l_1} (x_3^2-\gamma^2-|x|^2\sin^2\theta)^{-1/2}\sin\theta\, d\theta \leqslant \frac{t}{2\pi^2|x|}(1-m_1^2)^{-1/2}\int_0^{m_1}\frac{u\,du}{\sqrt{m_1^2-u^2}} = \frac{t}{2\pi^2 r} m_1 \leqslant \frac{t}{2\pi^2 r}$$

$$|J_3| \leqslant \frac{t}{2\pi^2}\int_{l_2}^{\pi/2} |x_3^2-\gamma^2-|x|^2\sin^2\theta|^{-1/2}\sin\theta\, d\theta \leqslant \frac{t}{2\pi^2|x|}\int_{m_2}^{1} [(u^2-m_2)(1-u^2)]^{-1/2} u\, du = \frac{t}{4\pi|x|}$$

and for estimating integral J_2 we use formula (3.5); then

$$|J_2| \leqslant \frac{t}{4\pi^3 r}\int_{l_1}^{l_2} d\theta \int_{-1}^{1} \frac{2r\sin\theta}{\gamma}\frac{d\tau}{\sqrt{1-\tau^2}} = \frac{t}{2\pi^2\gamma}[(1-m_1^2)^{1/2}-(1-m_2^2)^{1/2}] \leqslant \frac{t}{2\pi^2|x|}$$

From the derived estimates we obtain for functions $E_\gamma(x,t)$ the majorant

$$|E_\gamma(x,t)| \leqslant Ct/r, \quad C=\text{const} \tag{3.7}$$

which is valid for all $\gamma>0$ and all x that satisfy condition (3.2).

4. **Passage to limit in S' as $\gamma\to+0$.** Let us prove that the sought fundamental solution of the operator N, which is the limit of functions $E_\gamma(x,t)$ as $\gamma\to+0$ in the S' space, can be defined by formula

$$E(x,t) = -\frac{\theta(t)}{2\pi^2 N|x|}\int_{|x_3|/|x|}^{1} [(u^2-x_3^2/|x|^2)(1-u^2)]^{-1/2}\sin Ntu\, du \tag{4.1}$$

For this it is sufficient to prove that the sequence of functions $E_\gamma(x,t)$ converges to function $E(x,t)$ almost everywhere, as $\gamma\to+0$.

If the indicated convergence takes place, then, taking into account the presence in functions $E_\gamma(x,t)$ of the locally integrable majorant, it is possible to apply for any basic function $\varphi(x,t)\in S$ the Lebesgue theorem on the passing to limit in the integrand, according to which

$$\lim_{\gamma\to+0}\int_{R^4} E_\gamma(x,t)\varphi(x,t)\,dx\,dt = \int_{R^4} E(x,t)\varphi(x,t)\,dx\,dt$$

This equality by virtue of the local integrability of functions $E_\gamma(x,t)$ and $E(x,t)$ means that function $E(x,t)$ is the limit of functions $E_\gamma(x,t)$ in the space S', as $\gamma\to+0$.

We shall show that functions $E_\gamma(x,t)$ and $E(x,t)$ converge almost everywhere. We assume x and t to be fixed with $rx_3\neq 0$, and only consider values of γ that are smaller than $|x_3|/\sqrt{2}$.

Let us estimate $|E_\gamma(x,t)-E(x,t)|$. Using formula (3.4) we obtain

$$|E_\gamma - E| \leqslant \frac{\sqrt{2}}{4\pi^2 N|x|}\left\{\left|\int_0^{m_2} H(u)\frac{\sin Ntu}{\sqrt{1-u^2}}du\right| + \left|\int_{m_2}^{1}\left[H(u)-\frac{\sqrt{2}}{\sqrt{u^2-x_3^2/|x|^2}}\right]\frac{\sin Ntu}{\sqrt{1-u^2}}du\right|\right\} \tag{4.2}$$

We represent the integral over the interval $[0, m_2]$ as the sum of three integrals J_4, J_5, and J_6, respectively, over the intervals $[0, m_1(1-\eta_1)]$, $[m_1(1-\eta_1), m_1]$, and $[m_1, m_2]$, where η_1 is the positive number smaller than unity, selected below, and shall estimate each integral separately.

When estimating integral J_4 we take into account that in the integration interval the inequalities

$$|H| \leqslant |K|^{-1} (P - |K|)^{1/2} \leqslant 2^{-1/2} |L| |K|^{-3/2} \leqslant \sqrt{2}\, m_2 |x|^{-1} m_1^{-3} (2\eta_1 - \eta_1^2)^{-3/2} \sqrt{1-u^2}\gamma \leqslant 4|x| (2\eta_1 - \eta_1^2)^{-3/2} |x_3|^{-2} \gamma \sqrt{1-u^2}$$

are satisfied, and by virtue of these

$$|J_4| \leqslant \sqrt{2}\,\gamma (\pi |x_3|)^{-2} N^{-1} (2\eta_1 - \eta_1^2)^{-3/2} \int_0^{m_2} |\sin Ntu|\, du \leqslant \sqrt{2}\, N^{-1} \pi^{-2} |x|^{-1} (2\eta_1 - \eta_1^2)^{-3/2} |x_3|^{-1} \gamma \tag{4.3}$$

Note that in the case of integral J_5

$$|H| \leqslant |L|^{-1} (P - |K|)^{1/2} \leqslant (2|K|)^{-1/2} = 2^{-1/2} (m_1^2 - u^2)^{-1/2}$$

from which

$$|J_5| \leqslant \frac{t}{4\pi^2 r} \int_{m_1(1-\eta_1)}^{m_1} u (m_1^2 - u^2)^{-1/2} du \leqslant (2)^{-3/2} \pi^{-2} r^{-1} t m_2 (\eta_1)^{1/2} \leqslant (2)^{-3/2} t \pi^{-2} r^{-1} (\eta_1)^{1/2} \tag{4.4}$$

For estimating integral J_6 we use the first of the previously derived inequalities for J_2. We obtain

$$|J_6| \leqslant \frac{t}{2\pi^2 \gamma} [(1 - m_1^2)^{1/2} - (1 - m_2^2)^{1/2}] \leqslant \frac{t}{4\pi^2 r |x|} \gamma \tag{4.5}$$

We represent the integral in (4.2) over the interval $[m_2, 1]$ in the form of the sum of integrals J_7 and J_8 over the intervals $[m_2, m_2 + \eta_2]$ and $[m_2 + \eta_2, 1]$, respectively, where η_2 is the positive number smaller than unity, selected below.

We estimate integral J_7 taking into account the inequalities

$$|H| \leqslant \sqrt{2} P^{-1/2} \leqslant \sqrt{2} |K|^{-1/2} \leqslant \sqrt{2} (u^2 - m_2^2)^{-1/2}$$

We obtain

$$|J_7| \leqslant \frac{t}{\pi^2 r} \int_{m_2}^{m_2+\eta_2} u (u^2 - m_2^2)^{-1/2} du \leqslant \frac{t}{\pi^2 r} \sqrt{2\eta_2 m_2 + \eta_2^2} \leqslant \frac{t}{\pi^2 r} \sqrt{\eta_2} \tag{4.6}$$

When considering integral J_8 we use the inequalities

$$(u^2 - m_2^2)^{1/2} \geqslant \eta_2, \quad |K| \geqslant \eta_2^2, \quad P \geqslant \eta_2^2, \quad |L| \leqslant \frac{2\gamma}{|x|}, \quad |K| < 2$$

by virtue of which we successively have

$$|H - \sqrt{2} (u^2 - m_2^2)^{-1/2}| \leqslant \eta_2^{-3} |(P + K)^{1/2} (u^2 - m_2^2)^{1/2} - \sqrt{2} P| \leqslant 2^{-3/2} \eta_2^{-5} |(P + K)(K - \gamma^2 |x|^{-2}) - 2P^2| = 2^{-3/2} \eta_2^{-5} (3L_2^2 + 2\gamma^2 |x|^{-2} P) \leqslant 2^{-3/2} \eta_2^{-5} (12\gamma^2 |x|^{-2} + 2^{5/2} \gamma^2 |x|^{-2}) \leqslant 7\eta_2^{-5} |x|^{-2} \gamma^2$$

from which

$$|J_8| \leqslant 7 (2)^{-3/2} \eta_2^{-5} |x|^{-3} t \gamma^2 \pi^{-2} \int_{m_2+\eta_2}^{1} u (1 - u^2)^{-1/2} du \leqslant (2)^{3/2} \eta_2^{-5} |x|^{-3} t \gamma^2 \tag{4.7}$$

Assume now that (x, t) is a fixed point in the Euclidean space R^4 whose coordinates satisfy only the condition $x_3 r \neq 0$, being otherwise arbitrary; let also ε be an arbitrary positive number. Then, by virtue of (4.4) and (4.6), it is possible to find numbers η_1 and η_2 such that for any $\gamma \in [0, |x_3| / \sqrt{2}]$ the inequality

$$|J_5| + |J_7| < \varepsilon/2 \tag{4.8}$$

is satisfied.

By virtue of estimates (4.3), (4.5), and (4.7) it is possible to indicate for these numbers η_1 and η_2 such γ_0 that the inequality

$$|J_4| + |J_6| + |J_8| < \varepsilon/2, \quad \forall \gamma \in [0, \gamma_0] \tag{4.9}$$

is satisfied.

From (4.8), (4.9), and (4.4) follows that

$$| E_\gamma (x, t) - E (x, t) | < \varepsilon, \ \forall \gamma \in [0, \gamma_0]$$

which shows that as $\gamma \to +0$ the sequence of functions $E_\gamma (x, t)$ converges to $E (x, t)$ almost everywhere

5. The hydrodynamic meaning of the fundamental solution. It follows from (4.1) that the derived fundamental solution of the operator of internal waves has the following properties:

$$E(x,t) = \frac{\partial E (x, t)}{\partial t} = 0 \quad \text{for } t < 0$$

$$E(x,t) \to 0, \quad \frac{\partial E (x, t)}{\partial t} \to -\frac{1}{4\pi |x|} \quad \text{for } t \to +0 \text{ in } D'(R^3)$$

(D' is the space of generalized Sobolev— Schwartz functions which enable us to establish, with allowance for Eq.(1.2), clearly the hydrodynamic meaning of function $E (x, t)$).

Consider a continuously stratified fluid unbounded in all directions at rest at $t < 0$ whose density $\rho_0 (x_3)$ is distributed in conformity with the law described in Sect.1. Particles of the fluid are assumed to acquire at instant of time $t = 0$ velocities defined by vector $\mathbf{V} (x, 0)$ whose components along axes x_1, x_2, x_3 are

$$v_1 (x, 0) = \frac{x_1 x_3}{4\pi^2 r^2 |x|} - {}^1/_8 \operatorname{sgn} x_1 \cdot \delta (x_2) \cdot \operatorname{sgn} x_3, \quad v_2 (x, 0) = \frac{x_2 x_3}{4\pi r^2 |x|} - {}^1/_8 \delta (x_1) \cdot \operatorname{sgn} x_2 \cdot \operatorname{sgn} x_3, \quad v_2 (x, 0) = -\frac{1}{4\pi |x|}$$

This results in that at $t > 0$ internal waves begin to propagate in the fluid. This manifests itself in particular in that equal density surfaces (isopycs) cease to be horizontal planes.

In the linear theory and the Boussinesq approximation function $E (x, t)$ determines the isopyc's displacement at point x and instant of time t, while $\partial E (x, t)/\partial t$ determines the vertical velocity component $v_3 (x, t)$ of the fluid.

Using formula (4.1) we can observe the properties of internal waves at large dimensionless times Nt. Using conventional methods we derive from (4.1) for $Nt \to \infty$ the asymptotic formula

$$E (x, t) = -\frac{1}{(2\pi)^{3/2} Nr \sqrt{Nt}} \left[\frac{\sin (Nt |x_3|/|x| + {}^1/_4\pi)}{\sqrt{|x_3|/|x|}} + \sin (Nt - {}^1/_4\pi) + O\left(\frac{1}{(Nt)^{-1}}\right)\right] \tag{5.1}$$

which implies that we have a superposition of two types of waves.

First, there are standing waves defined by the second term in brackets. These waves are axially symmetric of infinite length and a frequency equal to the Brent— Viaisial frequency N, and with an amplitude decreasing with time as $1/\sqrt{Nt}$.

Then there are progressing waves defined by the first term in brackets in formula (5.1). These waves are also axially symmetric. Their surfaces of equal phase are conic surfaces $|x_3| / |x| = \text{const}$ whose angular velocity is $x_3/(rt)$ which decreases with time; at a fixed instant of time the maximum of its absolute value lies near the vertical x_3-axis, and the sign of its angular velocity coincides with the sign of x_3. With increasing time the number of such conic waves increases and their angular length correspondingly decreases.

These properties of waves are in accord with published laboratory experiments /7/ on inducing internal waves in a vessel containing linearly stratified fluid by imparting an initial velocity to fluid particles by a rapid movement over a short distance of a solid body of dimensions that are small in comparison with those of the vessel.

The amplitude of progressing waves decreases in the course of time as $1/\sqrt{Nt}$.

REFERENCES

1. SOBOLEV S.L., On a new problem of mathematical physics. Izv. Akad. Nauk. SSSR, Ser. Matem., Vol.18, No.1, 1954.

2. VISHIK M.I., The Cauchy problem for equations with operator coefficients, the mixed boundary value problem for systems of differential equations, and the approximate method of their solution. Matem. Sb., Vol.39, No.1, 1956.

3. GAL'PERIN S.A., The Cauchy problem of the Sobolev equation. Sib. Matem. Zh., Vol.4, No.4, 1963.

4. MASLENNIKOVA V.N., Estimate in L_p and the asymptotics as $t \to \infty$ of solution of the Cauchy problem for Sobolev's syatem. Tr. Matem. Inst. Akad. Nauk.SSSR, Vol.103, 1968.

5. SEKERZH-ZEN'KOVICH S.Ia.,The fundamental solution of the internal wave operator. Dokl. Akad. Nauk. SSSR, Vol.246, No.2, 1979.

6. VLADIMIROV V.S., Equations of Mathematical Physics. Moscow, NAUKA, 1971.

7. STEVENSON T.N., The phase configuration of internal waves around a body moving in a density stratified fluid. J. Fluid Mech., Vol.60, pt.4, 1973.

Translated by J.J.D.